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**GEOMETRIC NONLINEAR ANALYSIS OF STRUCTURES WITH  
NON-PRISMATIC MEMBERS RESTING ON ELASTIC  
FOUNDATION**

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**Abstract**

In this study, a theoretical analysis is presented to estimate the in-plane large displacement elastic stability behavior of structures having non-prismatic members of linearly and nonlinearly varying sections resting on elastic foundation (Winkler type) and subjected to static loads applied at joints only. The analysis adopts the beam-column approach and models the structural members as beam-column elements resting on distributed springs. The formulation of beam-column element is based on Euler approach allowing for the influence of the axial force on bending stiffness. Changes in member chord length due to axial deformation and flexural bowing are taken into account. The stability and bowing functions are estimated using methods of finite differences and finite segments. Also, approximate results have been obtained by using approximate stability and bowing functions of linearly and nonlinearly tapered members resting on elastic foundation. A computer program has been coded in QB language to carry out the proposed analysis of structures with prismatic or non-prismatic members of linearly and nonlinearly varying sections resting on elastic foundation. As a result of this study; the only difference between the analysis of non-prismatic members resting on elastic foundation and those which are not, when adopting the beam-column approach, is

represented in the stability and bowing functions, and this is reflected directly on the tangent stiffness matrix.

الخلاصة:

تتناول هذه الدراسة التحليل النظري للسلوكية المرنة للهياكل المستوية مع الأخذ بنظر الاعتبار الإزاحات الكبيرة الحاصلة فيها والتي تحتوي على أعضاء لا موشورية خطية ولا خطية التغير و مستقرة على أساس مرن من نوع (Winkler) ومعرضة إلى أحمال ساكنة مسلطة على المفاصل. تبنت هذه الدراسة طريقة العمود . العتبة وان اشتقاق عنصر العمود . العتبة قد تمّ بالاعتماد على طريقة ايلر ( Euler Approach ) . اخذ بنظر الاعتبار تأثير القوة المحورية على صلابة العزم وكذلك تأثير التغيرات في طول الوتر نتيجة الانفعال المحوري وتقوس الانحناء. تم اشتقاق دوال الاستقرار والانحناء باستعمال طريقتي الفروقات المحددة ( Finite Differences ) وطريقة القطع المحددة ( Finite Segments ) كما أمكن الحصول على دوال إستقرارية وانحناء تقريبية للأعضاء اللاموشورية الخطية واللاخطية التغير. تمت في هذه الدراسة كتابة برنامج بلغة بيسك السريعة لغرض استخدامه في تحليل المنشآت الحاوية على أعضاء لا موشورية (خطية ولا خطية التغير) مستقرة على أساس مرن. قورنت نتائج التحليل المستخرجة من الطرق أعلاه فيما بينها إذ تم الحصول على تقارب جيد بين النتائج المستحصلة من تلك الطرق. وقد تم التوصل إلى عدة استنتاجات مهمة منها أنّ الفرق الوحيد بين تحليل الأعضاء غيرالموشورية المستندة على أساس مرن وتلك التي لا تستند ، عند اعتماد الحل باستخدام طريقة العمود - العتبة، يأتي فقط من دوال الإستقرارية و الانحناء وهذا ينعكس مباشرة على مصفوفة الصلابة المماسية.

## **1. Introduction**

### **1.1 Elastic Foundation**

Many problems related to the structural engineering can be modeled by means of a beam or a beam-column on elastic foundation. Examples of these are railway tracks, footings and adjacent structural elements such as various types of stiffeners. The simple

common model to describe the elastic foundation is the Winkler model, which consists of infinitely number of linear springs defined by the foundation modulus ( $k$ ).

## **1.2 Geometric Nonlinearity**

It's meant that the structures can be analyzed for large displacements and elastic material properties. The effect of geometric nonlinearity may be divided into three categories: -

1. Change of member lateral stiffness (i.e. stability problem);
2. Change in member length due to bowing;
3. Large displacement problem.

### **1.2.1 Stability and Bowing Problems**

For the beam-column supported on elastic foundation (Winkler model) of constant soil stiffness, the modified stability functions for a prismatic beam-column were derived by Al-Sarraf as [1]:

$$M_1 = \frac{EI}{L} (S \theta_1 + SC \theta_2 + Q y_1 / L - qQ y_2 / L) \quad \dots(1-1)$$

$$M_2 = \frac{EI}{L} (SC \theta_1 + S \theta_2 + qQ y_1 / L - Q y_2 / L) \quad \dots(1-2)$$

$$V_1 = \frac{EI}{L} (Q \theta_1 + qQ \theta_2 + T y_1 / L - tT y_2 / L) \quad \dots(1-3)$$

$$V_2 = \frac{EI}{L} (qQ \theta_1 + Q \theta_2 + tT y_1 / L - T y_2 / L) \quad \dots(1-4)$$

Where

$M_1$  and  $M_2$  are end moments at ends 1 and 2 respectively

$V_1$  and  $V_2$  are end forces at ends 1 and 2 respectively

$S$  is the stiffness factor

$SC$  moment carry-over factor

$Q$  sway moment factor

qQ sway moment carry-over factor

T shear factor

tT shear carry-over factor

S, SC, Q, qQ, T, and tT are functions of the non-dimensional parameter:

$$\lambda L = \left( \frac{k}{4EI} \right)^{1/4} L \quad \dots(1-5)$$

and axial load parameter ( $\rho = P/PE$ ). Where k in Eq.(1-5) is the stiffness of the elastic foundation, which is equal to the modulus of subgrade reaction multiplied by the width of the beam-column. The relation between the axial deformation u and the axial force P can be expressed according to Oran [3]:

$$P = \frac{EA}{L}(u - C_b L) \quad \dots(1-6)$$

$$C_b = b_1 (\theta_1 + \theta_2)^2 + b_2 (\theta_1 - \theta_2)^2 \quad \dots(1-7)$$

is the length correction factor due to bowing action

$b_1$  and  $b_2$  : bowing functions

E: Modulus of elasticity

I: Moment of inertia,

A: Cross-sectional area, and

L: Initial (undeformed) member length.

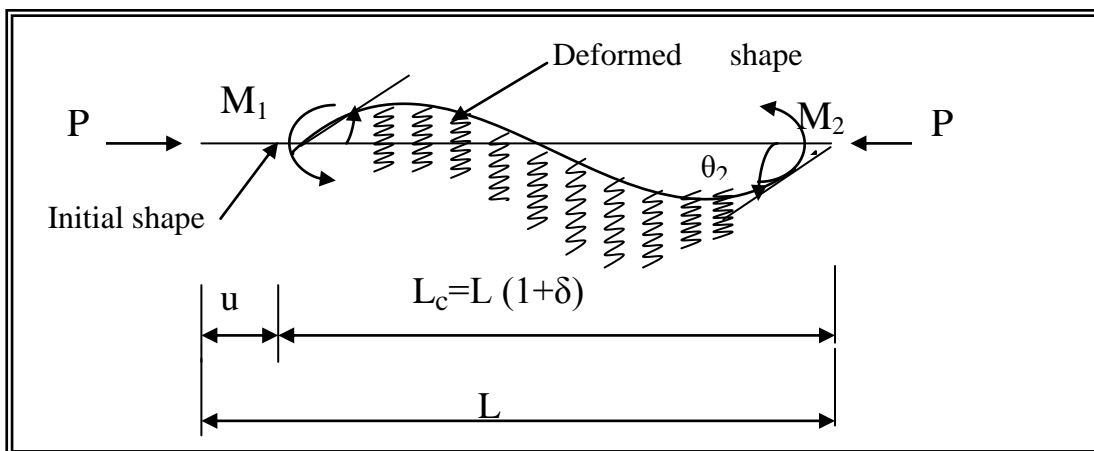


Fig.(1) Member forces and deformations in local coordinate

### **1.2.2 Large Displacement Problem**

The curvature of the beam-column element can be expressed as:

$$\Phi = \frac{1}{R} = \frac{-(d^2w/dx^2)}{[1 + (dw/dx)^2]^{1.5}} \quad \dots(1-8)$$

In small deformation theory, the term  $(dw/dx)$  has small values; therefore, the square of the term will be very small compared with one. Thus  $(dw/dx)$  is neglected. In large deformations, the slope  $(dw/dx)$  has a considerable effect on the curvature value and it cannot be neglected. If the term  $(dw/dx)$  in Eq.(1-8) is considered in the analysis, the analysis becomes so complicated that it can be solved only by using elliptical integrals. The complicated problem is then solved by considering relative small deformation in the member with large joint displacements, so, the small deformation theory is still applicable to the relative member deformations and the problem produced by assuming large joint displacement may then be solved by satisfying the equilibrium condition in the last (updated) configuration of the structure.

### **2. Non – Prismatic Beam-Column Resting on Elastic Foundation**

The basic differential equation for a non-prismatic beam-column on elastic foundation is[9]:

$$\frac{d^2}{dx^2} \left( EI_{(x)} \frac{d^2y}{dx^2} \right) + P \frac{d^2y}{dx^2} + k y = 0 \quad \dots(2-1)$$

Where  $y$  represents lateral deflection at distance  $x$  along the member,  $EI_{(x)}$  is the flexural stiffness of the member,  $P$  is axial force, and  $k$  represents the stiffness of the foundation. For non-prismatic member, the following representation can be considered as shown in Fig.(2) [2]. All the members considered have uniform taper in either one or two directions. Therefore, the depth  $d_x$  may be expressed by:

$$d_x = d_2 (x/a) \quad \dots(2-2)$$

The moment of inertia of the cross-sectional area of the member about the axis of buckling may be expressed in the form

$$I_{(x)} = I_2 (x/a)^m \quad \dots(2-3)$$

Where  $I_{(x)}$  is the moment of inertia at distance  $x$  from the origin  $O$ , and  $m$  is the shape factor that depends on the cross-sectional shape and dimensions of the member. The shape factor  $m$  may be evaluated by observing that Eq.(2-3) must give  $I_{(x)} = I_1$  when  $x = b$ . This condition yields the relation:

$$m = \log (I_1 / I_2) / \log U \quad \dots(2-4)$$

Where  $U$  is  $(d_1/d_2)$  end depth ratio. The value of shape factor  $m$  can be determined only when the dimensions of the cross-sections are known [4]. In the next item, stability and bowing functions will be determined from the solving of Eq.(2-1) by two methods of solution; finite difference and finite segments.

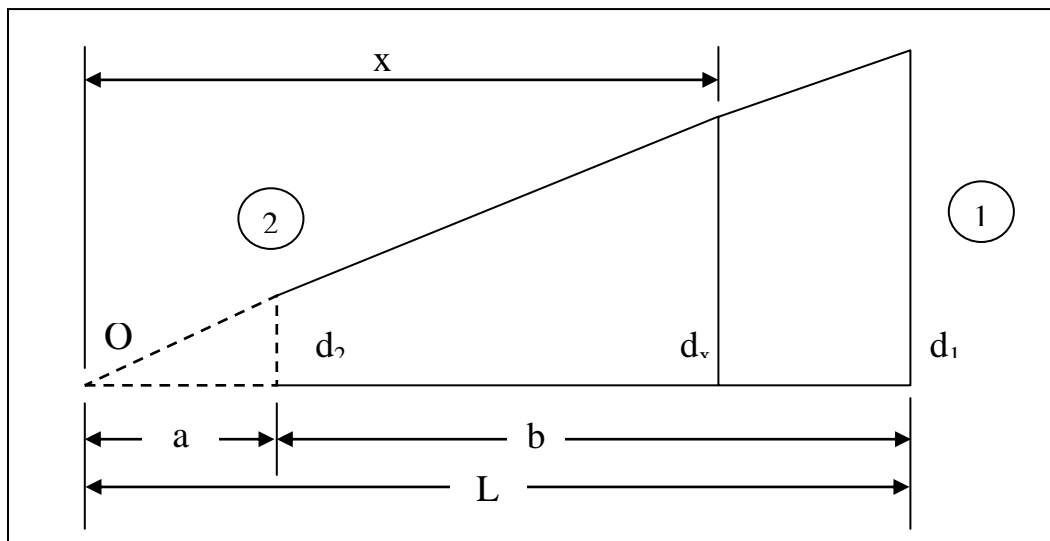


Fig.(2) Non-Prismatic Beam Coulmn

### **3. Stability and Bowing Problems**

#### **3.1 Modified Stability Functions**

##### **3.1.1 Estimation of Modified Stability Functions Using Finite Differences Method:**

In this method, the differential equation is replaced by a set of equivalent algebraic equations that are usually easier to solve than the differential equation. This method is

used to estimate the modified stability functions by solving the differential equation of the non-prismatic beam-column on elastic foundation (i.e. Eq.(2-1)). Ghali [5] presented the following formulation using the finite differences method for a beam resting on elastic foundation of variable section. The bending moment  $M$  and the deflection  $y$  are related by the differential equation:

$$EI_{(x)} \frac{d^2y}{dx^2} = -M \quad \dots(3-1)$$

by using central differences, the second derivative of lateral deflection is:

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \quad \dots(3-2)$$

where  $h = L / N$

in which:

$i$  number of the required node

$h$  interval between two successive nodes

$L$  length of the member

$N$  number of segments

By substituting the values of Eq.(3-2) into Eq.(3-1), the following is obtained

$$M_i = \frac{EI_i}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \quad \dots(3-3)$$

Here  $M_i$  is the moment at specified node  $i$ , and

$I_i$  is the moment of inertia at specified node  $i$

Double differentiation of Eq.(3-1) yields:

$$\frac{d^2}{dx^2} \left( EI_{(x)} \frac{d^2y}{dx^2} \right) = - \frac{d^2M}{dx^2} \quad \dots(3-4)$$

Then, the second derivative of the moment can be put in finite difference form,

$$\frac{d^2}{dx^2} \left( EI_{(x)} \frac{d^2y}{dx^2} \right) = - \frac{1}{h^2}(M_{i-1} - 2M_i + M_{i+1}) \quad \dots(3-5)$$

from Eq.(3-3),the right terms in Eq.(3-5) can be written as:

$$M_{i-1} = \frac{EI_{i-1}}{h^2} (y_{i-2} - 2y_{i-1} + y_i)$$

$$2M_i = \frac{2EI_i}{h^2} (y_{i-1} - 2y_i + y_{i+1}) \quad \dots(3-6)$$

$$M_{i+1} = \frac{EI_{i+1}}{h^2} (y_i - 2y_{i+1} + y_{i+2})$$

Substituting the above equations into Eq.(3-6), then sub Eqs.(3-6) and (3-2) into Eq.(2-1), yields the following:

$$\frac{-E}{h^4} \left[ I_{i-1} (y_{i-2} - 2y_{i-1} + y_i) - 2I_i (y_{i-1} - 2y_i + y_{i+1}) + I_{i+1} (y_i - 2y_{i+1} + y_{i+2}) \right] + \frac{P}{h^2} (y_{i-1} - 2y_i + y_{i+1}) + k y_i = 0 \quad \dots(3-7)$$

after re-arranging of Eq.(3-7),it becomes

$$I_{i-1} y_{i-2} + \left( \frac{Ph^2}{E} - 2(I_{i-1} + I_i) \right) y_{i-1} + \left( (I_{i-1} - 4I_i + I_{i+1}) \frac{2Ph^2}{E} + \frac{kh^4}{E} \right) y_i + \left( \frac{Ph^2}{E} - 2(I_{i+1} + I_i) \right) y_{i+1} + I_{i+1} y_{i+2} = 0 \quad \dots(3-8)$$

But  $h = L/N$  ;

$$\rho = P / Pe = PL^2 / \pi^2 EI_o \quad \dots(3-9)$$

and  $\lambda L = (k/4EI)^{1/4} .L$

Where  $I_o$  is the moment of inertia at node 0, (see Fig.(3)),Then, if the relations in Eq.(3-9) are substituted:

$$I_{i-1} y_{i-2} + A y_{i-1} + B y_i + C y_{i+1} + I_{i+1} y_{i+2} = 0 \quad \dots(3-10)$$



Where

$$A = (\rho I_o \pi^2 / N^2) - 2 (I_{i-1} + I_i) \quad \dots(3-11)$$

$$B = I_{i-1} + 4 I_i + I_{i+1} - (2\rho I_o \pi^2 / N^2) + ((\lambda L)^4 I_o / N^4) \quad \dots(3-12)$$

$$C = (\rho I_o \pi^2 / N^2) - 2 (I_i + I_{i+1}) \quad \dots(3-13)$$

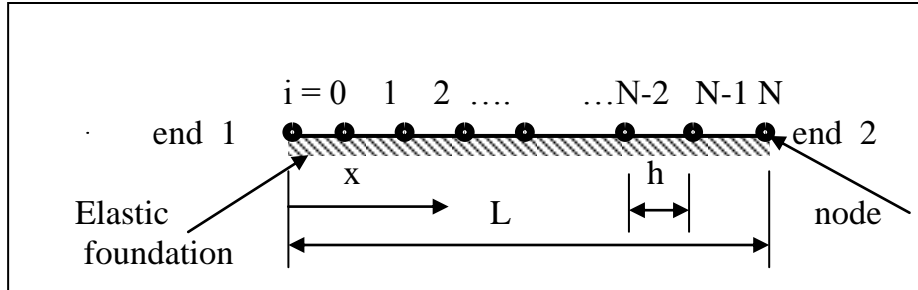


Fig.(3) Beam-column member resting on elastic foundation

Each stability function will be derived depending on the applied boundary conditions, for instance, the boundary conditions applied for determination of  $S_1$  and SC are: - ( $\theta_1 = 1, \theta_2 = y_0 = y_N = 0$ ).

**3.1.2 Estimation of Modified Stability Functions Using Finite Segments Method**

Finite segment method may be considered as a physical interpretation of the finite difference method that can be applied numerically to solve differential equations [7]. In this method, the non-prismatic member on elastic foundation is divided into (n) prismatic members, as shown in Fig.(4). The exact stability functions derived by Al-Hachami [6], which shown in chapter three, are used to calculate the modified stability functions.

For the segment m, the local end force-deformation relationships are:

$$M_i = \frac{EI_i}{h_m} \left[ C_{1m}\phi_i + C_{2m}\phi_j + (C_{1m} + C_{2m}) \frac{y_i}{h_m} - (C_{1m} + C_{2m}) \frac{y_j}{h_m} \right] \quad \dots (3-14)$$

$$M_j = \frac{EI_i}{h_m} \left[ C_{2m}\phi_i + C_{1m}\phi_j + (C_{1m} + C_{2m}) \frac{y_i}{h_m} - (C_{1m} + C_{2m}) \frac{y_j}{h_m} \right] \quad \dots (3-15)$$

$$V_i = \frac{EI_i}{h_m^2} \left[ (C_{1m} + C_{2m})\phi_i + (C_{1m} + C_{2m})\phi_j + A_m \frac{y_i}{h_m} - A_m \frac{y_j}{h_m} \right] \dots(316)$$

$$V_j = \frac{EI_i}{h_m^2} \left[ -(C_{1m} + C_{2m})\phi_i - (C_{1m} + C_{2m})\phi_j - A_m \frac{y_i}{h_m} - A_m \frac{y_j}{h_m} \right] \dots(3-17)$$

Equations (3-14) to (3-17) can be written in matrix form as:

$$\{f\}_m = [K]_m \{V\}_m \dots(3-18)$$

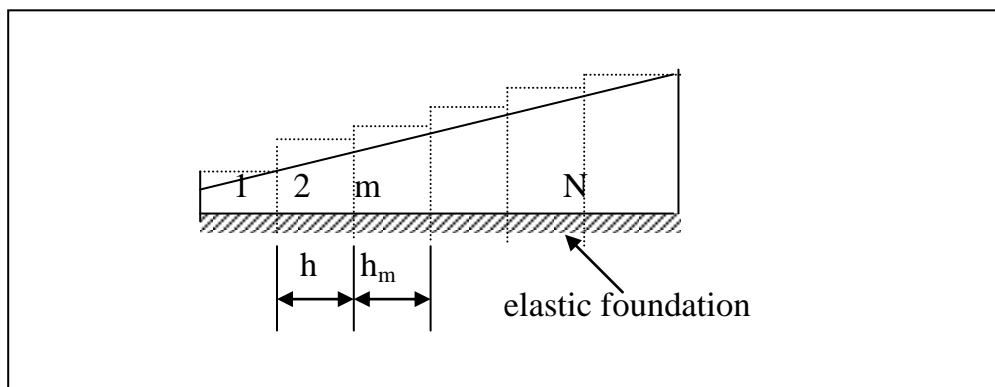


Fig.(4) Member segments of non-prismatic beam-column on elastic Foundation

$$\{f\}_m = \begin{Bmatrix} V_i L \\ M_i \\ V_j L \\ M_j \end{Bmatrix} \dots(3-19)$$

$$\{V\}_m = \begin{Bmatrix} y_i / L \\ \phi_i \\ y_j / L \\ \phi_j \end{Bmatrix} \dots(3-20)$$

and the stiffness matrix  $[K]_m$  can be written as:

$$[K]_m = \frac{EI_m}{L} \begin{bmatrix} \frac{A_m}{f_{rm}^3} & \frac{(C_{1m} + C_{2m})}{f_{rm}^2} & -\frac{A_m}{f_{rm}^3} & \frac{(C_{1m} + C_{2m})}{f_{rm}^2} \\ & \frac{C_{1m}}{f_{rm}} & \frac{(C_{1m} + C_{2m})}{f_{rm}^2} & \frac{C_{2m}}{f_{rm}} \\ & & \frac{A_m}{f_{rm}^3} & -\frac{(C_{1m} + C_{2m})}{f_{rm}^2} \\ & & & \frac{C_{1m}}{f_{rm}} \end{bmatrix} \quad ..(3-21)$$

In which:

$$A_m = 2(C_{1m} + C_{2m}) - \pi^2 q_m \quad \dots(3-22)$$

$$q_m = q_e \cdot f_{rm}^2 \quad \dots(3-23)$$

Where  $q_m$  is the segment m axial force parameter, while  $q_e$  is the total element axial force parameter

$$f_{rm} = h_m / L \quad \dots(3-24)$$

$C_{1m}$ , and  $C_{2m}$ : stability functions of a prismatic segment, which are functions of  $q_m$ .

$y_i$  and  $y_j$ : are sways of end i and j of segment m.

$\varphi_i$  and  $\varphi_j$ : are angles of rotations of end i and j of sement m.

$h_m$ : is the length of segment m.

$I_m$ : is the moment of inertia for segment m.

For the case of beam-column resting on elastic foundation (Winkler model), where the soil subgrade reaction is assumed to be uniformly distributed along the beam-column, the segment stiffness matrix  $[K]_m$  in Eq.(3-21) must be rewritten as [6]:

$$[K]_m = \frac{EI_m}{L} \begin{bmatrix} \frac{J_{3m}}{f_{rm}^3} & \frac{J_{1m}}{f_{rm}^2} & -\frac{J_{4m}}{f_{rm}^3} & \frac{J_{2m}}{f_{rm}^2} \\ \frac{C_{1m}}{f_{rm}} & \frac{J_{2m}}{f_{rm}} & -\frac{J_{3m}}{f_{rm}^2} & \frac{C_{2m}}{f_{rm}} \\ \frac{J_{3m}}{f_{rm}^3} & -\frac{J_{1m}}{f_{rm}^2} & \frac{J_{4m}}{f_{rm}^3} & -\frac{J_{2m}}{f_{rm}^2} \\ \frac{C_{1m}}{f_{rm}} & \frac{J_{2m}}{f_{rm}} & -\frac{J_{3m}}{f_{rm}^2} & \frac{C_{2m}}{f_{rm}} \end{bmatrix} \dots(3-25)$$

where  $C_{1m}, C_{2m}, J_{1m}, J_{2m}, J_{3m}$ , and  $J_{4m}$  denote the stability functions for a prismatic beam-column resting on elastic foundation.

$$(\lambda L)_m = f_{rm} \cdot (\lambda L)_e \dots(3-26)$$

where  $(\lambda L)_m$  is the segment m axial force parameter, while  $(\lambda L)_e$  is the total element axial force parameter. Each stability function will be derived depending on the applied boundary conditions.

### **3.2 Modified Bowing Functions**

Oran [8] discovered that the modified stability functions ( $\gamma_1, \gamma_2$ , and  $\gamma_3$ ) can be found by using the stability functions ( $S_1, SC, S_2$ ), as follows:

$$\gamma_1 = \frac{S_2}{U^m} \dots(3-27)$$

$$\gamma_2 = \frac{SC}{U^m} \dots(3-28)$$

$$\gamma_3 = \frac{S_1}{U^m} \dots(3-29)$$

Also, he derived the modified bowing functions for the non-prismatic members, as follows:

$$\beta_i = \frac{-\gamma'_i}{2\pi^2} \dots(3-30)$$

$$\beta'_i = \frac{-\gamma''_i}{2\pi^2} \dots(3-31)$$

Where  $\beta_i$  = modified bowing function.

$\beta'_i$  = first derivative of bowing function.

$\gamma'_i$  = first derivative of stability function.

$\gamma''_i$  = second derivative of stability functions.

$\gamma'_i$  can be found using the finite difference method.

$$f'(q) = \frac{f(q + \Delta q) - f(q - \Delta q)}{2\Delta q} \quad \dots(3-32)$$

$$f''(q) = \frac{f(q + \Delta q) - 2f(q) + f(q - \Delta q)}{(\Delta q)^2} \quad \dots(3-33)$$

### **3.3 Estimation of Modified Stability and Bowing Functions Using Approximate Formulae**

#### **3.3.1 Estimation of Stability Functions**

In order to facilitate the estimation of modified stability functions of the non-prismatic members resting on elastic foundation, and depending on the results obtained previously by finite differences method and finite segments method, it is noticed that the modified stability functions for members with linear and non-linear variation of sections can be estimated using the relations:

$$S_1 = U^{(\varphi - 0.1)m/4} .S \quad \dots(3-34)$$

$$\overline{SC} = U^{(1 + \varphi)m/4.3} .SC \quad \dots(3-35)$$

$$S_2 = U^{(\varphi + 0.1)m/1.47} .S \quad \dots(3-36)$$

$$Q_1 = U^{\varphi m/3.85} .Q \quad (\text{for } \lambda L < 3) \quad \dots(3-37)$$

$$Q_1 = U^{(\varphi m/1.7\lambda L)} .Q \quad (\text{for } \lambda L \geq 3) \quad \dots(3-38)$$

$$Q_2 = U^{\varphi m/(1.6 - 0.1\lambda L)} .Q \quad (\text{for } \lambda L \leq 4.5) \quad \dots(3-39)$$

$$Q_2 = U^{\varphi m/(1.195 - 0.01\lambda L)} .Q \quad (\text{for } \lambda L > 4.5) \quad \dots(3-40)$$

$$qQ_1 = U^{\varphi m/1.4} .qQ \quad \dots(3-41)$$

$$qQ_2 = U^{\varphi m/(\lambda L + 1)} .qQ \quad (\text{for } \lambda L < 3) \quad \dots(3-42)$$

$$qQ_2 = U^{\varphi m/3.85} .qQ \quad (\text{for } \lambda L \geq 3) \quad \dots(3-43)$$

$$T_1 = U^{\varphi m/(3.5 - (0.25 - 0.1 U^{(U-1)(U-1.5)}))} .T \quad (\text{for } \lambda L \leq 1.5) \quad \dots(3-44)$$

$$T_1 = U^{\phi m / (2.8 - 0.25 (U - 1.5) + 0.5 (\lambda L - 2))} \cdot T \quad (\text{for } 1.5 < \lambda L < 4) \quad \dots(3-45)$$

$$T_1 = U^{\phi m / ((\lambda L - 1) + 0.6 - 0.2 (\lambda L - 4))} \cdot T \quad (\text{for } \lambda L \geq 4) \quad \dots(3-46)$$

$$T_2 = U^{\phi m / (1.98 - 0.03 (U - 1.5))} \cdot T \quad (\text{for } \lambda L \leq 1.5) \quad \dots(3-47)$$

$$T_2 = U^{\phi m / (1.83 - 0.183 (U - 1.5) - (0.25 - 0.115 (U - 1.5)(U - .5) (\lambda L - 2)))} \cdot T \quad (\text{for } 1.5 < \lambda L < 4) \quad \dots(3-48)$$

$$T_2 = U^{\phi m / (1.43 - 0.1(\lambda L - 4) - .01 U)} \cdot T \quad (\text{for } \lambda L \geq 4) \quad \dots(3-49)$$

$$tT_1 = U^{\phi m / 2} \cdot tT \quad \dots(3-50)$$

$$tT_2 = U^{\phi m (1 + (\lambda L - 1) / 7) / (5.3 - 0.9 U)} \cdot tT \quad \dots(3-51)$$

Where S, SC, Q, qQ, T, tT are the stability functions of a uniform member resting on elastic foundation with constant moment of inertia ( $U^{\phi m / 2} \cdot I_1$ ), having axial load parameter

$$\rho = \rho_1 / U^{\phi m / 2} \quad \dots(3-52)$$

The value of  $\phi$  depends on the shape factor m [2]. As follows:

A- For members having linear distribution of cross section (i.e. tapered member).

$$\text{For } m = 4 \quad \phi = 1$$

$$\text{For } m \leq 3 \quad \phi = 1.04 + 0.08 (3 - m) \quad \dots(3-53)$$

B- For members having parabolic distribution of cross section, it is divided into:

1-members having concave variation of cross section [7]

$$\text{For } m = 4 \quad \phi = 0.825$$

$$\text{For } m \leq 3 \quad \phi = 0.88 - 0.284 (3 - m) \quad \dots(3-54)$$

2-members having convex variation of cross section

$$\text{For } m = 4 \quad \phi = 1.07$$

$$\text{For } m \leq 3 \quad \phi = 1.09 + 0.08(3 - m) \quad \dots(3-55)$$

### **3.3.2 Estimation of Bowing Functions**

In connection with approximate stability function, the values of  $\gamma_i$  are:

$$\gamma_1 = U^{(\phi - 1.37) m / 1.47} \cdot S \quad \dots(3-56)$$

$$\gamma_2 = U^{(\phi - 3.3) m / 4.3} \cdot SC \quad \dots(3-57)$$

$$\gamma_3 = U^{(\phi - 4.1) m / 4} \cdot S \quad \dots(3-58)$$

The derivatives of the stability functions  $\gamma_i$  with respect to the axial load parameter ( $\rho_1$ ):

$$\gamma'_1 = U^{(0.2625 \phi + 0.1) m / 1.47} \cdot S' \quad \dots(3-59)$$

$$\gamma'_2 = U^{(1-1.15\phi)m/4.3} \cdot SC' \quad \dots(3-60)$$

$$\gamma'_3 = U^{-(\phi+0.1)m/4} \cdot S' \quad \dots(3-61)$$

$$\gamma''_1 = U^{(1.57-0.47\phi)m/1.47} \cdot S'' \quad \dots(3-62)$$

$$\gamma''_2 = U^{(5.3-3.3\phi)m/4.3} \cdot SC'' \quad \dots(3-63)$$

$$\gamma''_3 = U^{-(4.1+3\phi)m/4} \cdot S'' \quad \dots(3-64)$$

Where  $S'$ ,  $SC'$ ,  $S''$ , and  $SC''$  can be determined approximately by using finite difference method from Eqs.( 3-32) and (3-33). Now, bowing function ( $\beta_i$ ) can be obtained with respect to ( $\rho_1$ ) using:

$$\beta_i = \frac{-\gamma'_i}{2\pi^2} \quad \dots(3-65)$$

Graphs of stability functions for various types of non-prismatic members are shown in appendix A. In addition, results of bowing functions is illustrated in Tables (A-1), (A-2) and (A-3) in appendix A, also.

#### **4. Modified Tangent Stiffness Matrix in Local Coordinates**

For convenience, the following notation is introduced [8]

$$U_1 = \theta_1 \quad ; \quad U_2 = \theta_2 \quad ; \quad U_3 = u \quad \dots(3-66)$$

and

$$\bar{S}_1 = M_1 \quad ; \quad \bar{S}_2 = M_2 \quad ; \quad \bar{S}_3 = PL \quad \dots(3-67)$$

$$\text{then , } \{\bar{\Delta S}\} = [t] \{\Delta U\} \quad \dots(3-68)$$

in which  $[t]$  = tangent stiffness matrix for relative deformation, with

$$t_{ij} = \frac{\partial S_i}{\partial U_j} + \frac{\partial S_i}{\partial \rho} * \frac{\partial \rho_1}{\partial U_j} \quad ; \quad \text{for } i, j = 1, 2, 3 \quad \dots(3-69)$$

$\rho_1$  can be rewritten in the form:

$$\rho_1 = \frac{\lambda^2}{\pi^2} (\text{ur} - (\beta_1 \theta_1^2 + 2\beta_2 \theta_1 \theta_2 + \beta_3 \theta_2^2)) \quad \dots(3-70)$$

Using a notation consistent to that of a prismatic member [3,8]

$$\partial \rho / \partial \theta_1 = G_1 / \pi^2 H; \quad \partial \rho / \partial \theta_2 = G_2 / \pi^2 H; \quad \partial \rho / \partial U = 1/H \quad \dots(3-71)$$

$$\text{in which } \text{ur} = u / L \quad \dots(3-72)$$

$$G_1 = -2\pi^2 (\beta_1 \theta_1 + \beta_2 \theta_2) = \gamma'_1 \theta_1 + \gamma'_2 \theta_2 \quad \dots(3-73)$$

$$G_2 = -2\pi^2 (\beta_2 \theta_1 + \beta_3 \theta_2) = \gamma'_2 \theta_1 + \gamma'_3 \theta_2 \quad \dots(3-74)$$

$$H = \frac{\pi^2}{\lambda^2} + (\beta'_1 \theta_1^2 + 2\beta'_2 \theta_1 \theta_2 + \beta'_3 \theta_2^2) \quad \dots(3-75)$$

The tangent stiffness matrix is:

$$[t] = (EI_1 / L) \begin{bmatrix} \gamma_1 + \frac{G_1^2}{\pi^2 H} & \gamma_2 + \frac{G_1 G_2}{\pi^2 H} & \frac{G_1}{H} \\ & \gamma_3 + \frac{G_2^2}{\pi^2 H} & \frac{G_2}{H} \\ \text{Sym} & & \frac{\pi^2}{H} \end{bmatrix} \quad \dots(3-76)$$

## **4.Applications**

### **4.1 Introduction**

The nonlinear response of structures is generated by an incremental load approach with Newton – Raphson type of iteration performed at each load increment to satisfy the joint equilibrium equations. A computer program called (ELDATFEF) is written to carry out the large displacement elastic stability analysis of plane frames comprised of prismatic or non-prismatic (tapered and non-uniform) members resting on elastic foundation (Winkler model). The computer program is coded in (Quick Basic Ver.4.5) language to be used on IBM.PC computers. The flowchart of this program with a brief description for each subroutine is shown in appendix B.



**4.2 Examples**

**Example 1: Finite Beam Subjected to End Load**

A beam of 60 ft (18.288 m) length rests on an elastic foundation. A concentrated load of 50 kips (222.5 kN) is applied at one – end as shown in Fig.(5-1-a). The other properties are: area = 23.54 in<sup>2</sup> (0.0152 m<sup>2</sup>) , moment of inertia = 1047.08 in<sup>4</sup> (2 \* 10<sup>-7</sup> m<sup>4</sup>) and modulus of elasticity = 30000 kips/in<sup>2</sup> (206850 N/mm<sup>2</sup>). Al-Hachami [6] solved this example for small values of ( $\lambda L$ ), and as he stated, end B must be supported against both vertical deflection and rotation, otherwise no solution can be obtained. He used six elements in solving this problem, while in the present study one element is used to get good agreement between the present study and that mentioned previously, Fig.(5-1-b).

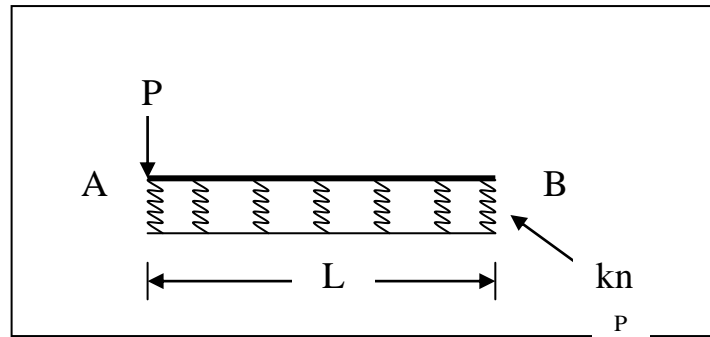
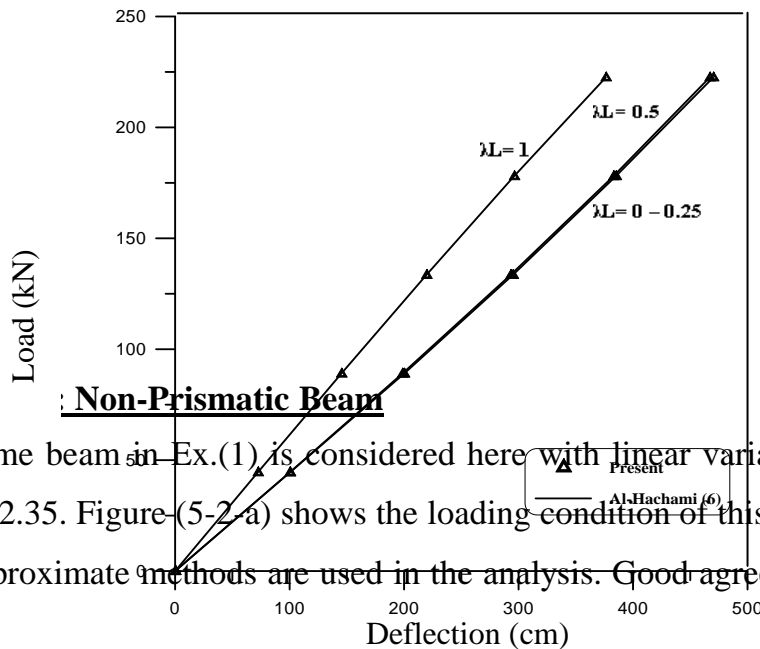


Fig.(5-1-a) Example 1



**Exam : Non-Prismatic Beam**

The same beam in Ex.(1) is considered here with linear variation of section. End depth ratio = 2.35. Figure (5-2-a) shows the loading condition of this example. Finite difference and approximate methods are used in the analysis. Good agreement can be seen between

Fig.(5-1-b) Graph of Example 1

the results obtained from the two methods of analysis. Figure (5-2-b) shows the load-displacement curves of this example for  $\lambda L = 1$ .

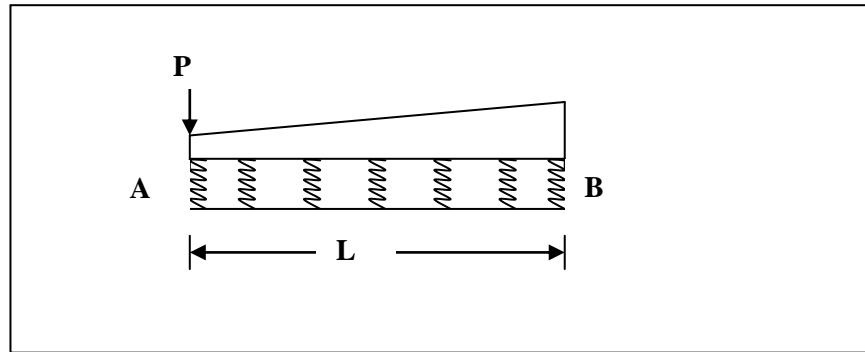


Fig.(5-2-a) Example 2

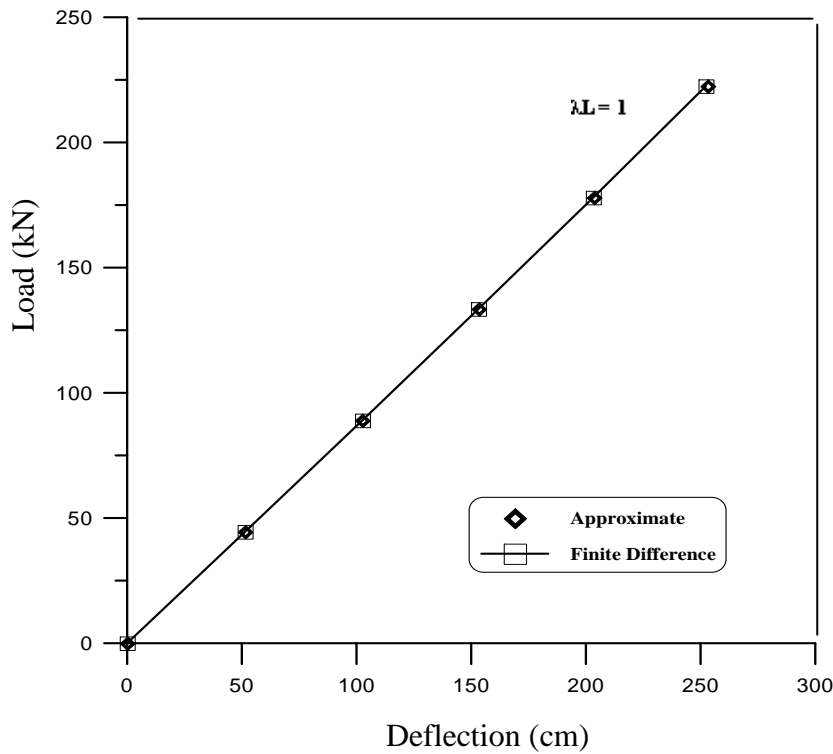


Fig.(5-2-b) Graph of Example 2

## **6. Conclusions**

1. The stability and bowing functions can be derived using finite difference method and finite segment method.
2. For linearly and nonlinearly tapered members resting on elastic foundation, stability and bowing functions can be estimated approximately by using the stability and bowing functions for prismatic members using different factors depending on the tapering ratio, shape factor, axial force parameter and sometimes non-dimensional soil parameter.
3. The only difference between the analysis of non-prismatic members resting on elastic foundation and those which are not, when adopting the beam-column approach, is represented in the stability and bowing functions, and this is reflected directly on the tangent stiffness matrix.

## **References**

1. Al-Sarraf, S.Z., "Elastic Instability of Struts on, or Driven into, Elastic Foundation", *The Structural Engineer*, Vol.56B, No.1, March, 1978, pp.13-19.
2. Al-Sarraf, S.Z., "Elastic Instability of Frames with Uniformly Tapered Members", *The Structural Engineer*, Vol.57B, No.1, March, 1979, pp.18-24
3. Oran, C., "Tangent Stiffness in Plane Frames", *Journal of the Structural Division*, ASCE, Vol.99, No.ST6, June, 1973, pp.973-985.
4. Gupta, A.K., "Vibration of Tapered Beams", *Journal of Structural Engineering*, ASCE, Vol.111, No.1, Jan., 1985, pp.19-36.
5. Ghali, A. and Neville, A.M., "Structural Analysis", E and FN Spon, London, 4<sup>th</sup> Edition, 1997, pp.437-446.
6. Al-Hachami, E.K., "Large Displacement Analysis of Structures with Applications to Piles and Submarine Pipelines", Ph.D. thesis, University of Technology – Iraq, 1997.

7. Faris, H.A.,” Large Displacement Elastic-Plastic Analysis of Plane Frames with Non-Prismatic Members of Non-linearly Varying Sections”, Ph.D. thesis, University of Technology – Iraq, 2002.
8. Oran, C.,”Geometric Nonlinearity in Nonprismatic Members”, Journal of the Structural Division, ASCE, Vol.100, No.ST7, July, 1974,pp.1473-1487.
9. Al-Ejbari,A.T.,”Large Displacement Elastic Stability Analysis of Plane Frames with Non-Prismatic Members Resting on Elastic foundation”,Ms.thesis, Al-Anbar University – Iraq ,2004

### Nomenclature

$A_o$	Equivalent cross – sectional area
$a$	$L / (U - 1)$
$b_1, b_2$	Bowing functions for prismatic members
$C_b$	Length correction factor due to bowing
$[d]$	Element nodal displacements
$d_i$	Depth of cross – section at end i
$\{F\}$	Vector of member end forces in global coordinates
$[f]$	Element nodal forces
$G_1, G_2, H$	Nonlinear geometric functions
$I_{(x)}$	Variable moment of inertia of the beam-column member
$I_i$	Moment of inertia at end i
$k$	Stiffness of the elastic foundation
$k_s$	Modulus of subgrade reaction
$L$	Initial (undeformed) member chord length
$L_c$	Chord length of deformed member
$m$	Shape factor = $\text{Log}(I_1/I_2)/\text{Log}(U)$
$P$	Member axial force
$P_E$	Classical Euler buckling load for a beam-column = $\pi^2 EI/L^2$
$\{S\}$	Vector of member end forces in Eulerian coordinates
$[T]$	Member tangent stiffness matrix in global coordinates
$[t]$	Member tangent stiffness matrix in local coordinates
$U$	Tapering ratio = $d_1/d_2$
$\{U\}$	Vector of member end displacements in global coordinates
$X, Y$	Initial joint global coordinates
$\beta_i$	Bowing functions for non-prismatic members
$\gamma_i$	Stability functions for non-prismatic member
$\theta_1$	Relative rotation of end 1 of beam-column element
$\theta_2$	Relative rotation of end 2 of beam-column element
$\Delta$	Increment
$[\tau]$	System (structural) assembled tangent stiffness matrix